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Quantum parameter space in super Yang-Mills, II

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In [1] (hep-th/0211069), the author has discussed the quantum parameter space of the $\mathcal{N} = 1$ super Yang-Mills theory with one adjoint Higgs field Φ, tree-level superpotential $W_{\rm tree} = m\Phi^2/2 + g\Phi^3/3$, and gauge group $\mathrm{U}(N)$. In particular, full details were worked out for U(2) and U(3). By discussing higher rank gauge groups like U(4), for which the classical parameter space has a large number of disconnected components, we show that the phenomena discussed in [1] are generic. It turns out that the quantum space is connected. The classical components are related in the quantum theory either through standard singularities with massless monopoles or by branch cuts without going through any singularity. The branching points associated with the branch cuts correspond to new strong coupling singularities, which are not associated with vanishing cycles in the geometry, and at which glueballs can become massless. The transitions discussed recently by Cachazo, Seiberg and Witten are special instances of those phenomena.

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1 Introduction and review of U(2) and U(3)

In a recent paper [1], the powerful technology in the calculation of exact quantum effective superpotentials [2, 3, 4, 5] was used for the first time to derive new physics in $\mathcal{N}=1$ supersymmetric $\mathrm{U}(N)$ gauge theories. The basic object considered in [1] is the quantum space of parameters $\mathcal{M}_{\mathrm{q}}$. This space is reminiscent of the quantum moduli space of theories with a larger number of supersymmetries. The most fundamental difference is that no massless scalar is associated with the motion on $\mathcal{M}_{\mathrm{q}}$. The problem of calculating $\mathcal{M}_{\mathrm{q}}$ is thus very general and also occurs in non-supersymmetric theories, as exemplified in [6]. The space $\mathcal{M}_{\mathrm{q}}$ describes the phase diagram as well as some non-trivial phenomena that occur in given phases.

As in [1], the example that we consider is the $\mathcal{N}=1$ theory with gauge group $\mathrm{U}(N)$ and one adjoint Higgs field Φ with tree-level superpotential

$$W_{\mathrm{tree}}(\Phi)=\frac{m}{2}\,\Phi^2+\frac{g}{3}\,\Phi^3\,. \tag{1.1}$$

Quantum mechanically, the theory depends on a single dimensionless parameter

$$\lambda=\frac{8g^2\Lambda^2}{m^2}\,, \tag{1.2}$$

where Λ is the dynamically generated scale that governs the UV running of the gauge coupling constant. Weak coupling corresponds to $\lambda\rightarrow 0$. In the small λ region, the parameter space is simply the union of several disconnected components, or sheets, associated with the various classical vacua. For example, the U(2) theory has five disconnected components by taking into account chiral symmetry breaking in the low energy gauge group. Two vacua correspond to a classically unbroken gauge group U(2) with $\langle\phi\rangle_{\mathrm{cl}}=0$, two others to similar vacua with $\langle\phi\rangle_{\mathrm{cl}}=-m/g$, and a last vacuum corresponds to a classically unbroken gauge group $\mathrm{U}(1)\times\mathrm{U}(1)$. Similarly, the U(3) theory has ten weakly coupled components, and the U(4) theory eighteen. For $\mathrm{U}(N)$, there are $N(N^2+11)/6$ components that can be labeled as $|k_1,k_2;N_1,N_2\rangle$. The integers N_1 and N_2, $N_1+N_2=N$, give the number of eigenvalues of $\langle\phi\rangle_{\mathrm{cl}}$ at zero and $-m/g$ respectively, corresponding to a classical breaking of the gauge group from $\mathrm{U}(N)$ down to $\mathrm{U}(N_1)\times\mathrm{U}(N_2)$. The integers k_j, defined modulo N_j and usually chosen such that $0\leq k_j\leq N_j-1$, label the various chirally asymmetric vacua of the low energy theory. Note that if the model were purely classical, any two sheets of the parameter space would meet only at $\lambda=\infty$ or equivalently for a critical $m=0$ tree-level superpotential (1.1).

The full quantum parameter space for U(2) was drawn in the figure 3 of [1]. All the relevant calculations for U(3) were also included in [1], and we have depicted the

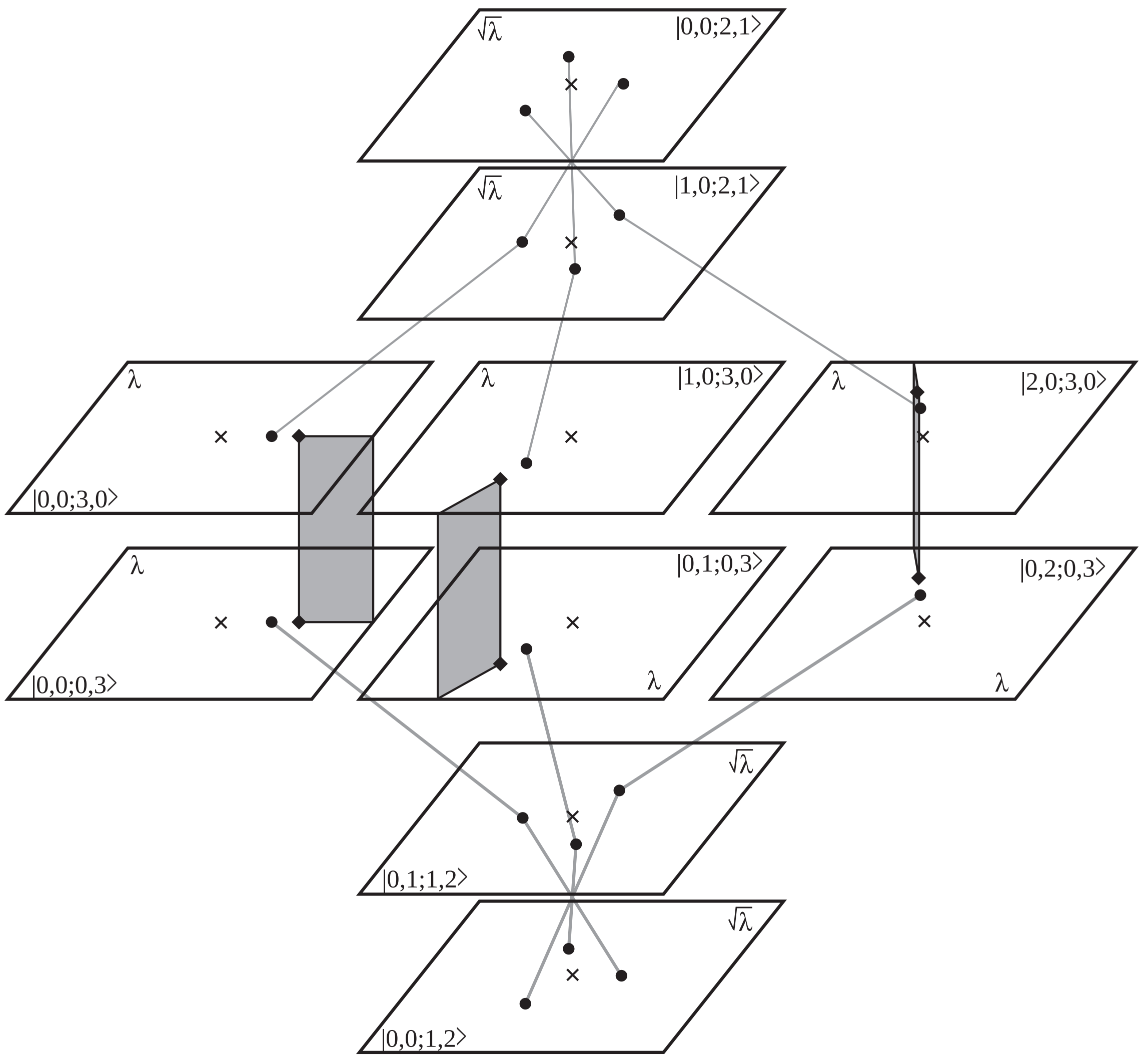

Figure 1: Sketch of the quantum parameter space $\mathcal{M}_{\mathrm{q}}$ for gauge group U(3). The sheets are labeled by a state $|k_1, k_2; N_1, N_2\rangle$ and parametrized by λ (if $N_1 N_2 = 0$) or by $\sqrt{\lambda}$ (if $N_1 N_2 \neq 0$). The cross denotes the classical $\lambda = 0$ points. The black dots represent singularities with a massless monopole that are found for $\sqrt{\lambda} = 2\sqrt{2} e^{-i\pi q/3}/3$, q integer. Each singularity of that type link three sheets, and the gray lines represent the corresponding three-fold indentification. The black squares at $\lambda = e^{-2i\pi k/3}$ represent singularities with a massless glueball. They are the three branching points of the three cuts joining the sheets $|k, 0; 3, 0\rangle$ and $|0, k; 0, 3\rangle$.

resulting $\mathcal{M}_{\mathrm{q}}$ in figure 1. A most notable feature is that the quantum parameter space is connected, and this is probably the case for any gauge group $\mathrm{U}(N)$. This connectedness results from two basic mechanisms described in [1].

The first mechanism corresponds to a phase transition with massless monopole(s) and vanishing string tension(s). For example, a Coulomb phase and a confining phase are related in that way in figure 1 (gray lines). Such transitions come with vanishing cycles in the geometry, yielding a singular matrix model curve in the approach of [4], or equivalently a singular Calabi-Yau in the string theory approach of [2]. Those singularities are reminiscent of the singularities on the moduli space of $\mathcal{N}=2$ theories.

The second mechanism involves the presence of branch cuts in parameter space whose origin can be easily described. The expectation values of gauge invariant chiral operators, like the $u_k = \operatorname{tr}\phi^k/N$, or the glueball superfields S_i that are the basic variables in the string ot matrix model approaches [2, 4], are generically analytic functions of the parameters. A particularly useful example for us is

$$u = -\frac{g}{m}\,\langle \operatorname{tr}\phi\rangle\,, \tag{1.3}$$

whose classical value in a vacuum $|k_1,k_2;N_1,N_2\rangle$ is simply

$$u_{\mathrm{cl}} = u(\lambda=0) = N_2\,. \tag{1.4}$$

The vevs are found by extremizing suitable effective quantum superpotentials, and this amounts to solving an algebraic equation. One then gets infinite fractional instanton series that have a finite radius of convergence. The corresponding analytic functions have branch cuts. For example, in the $\mathrm{U}(N)$ theory, one has [1]

$$u(\lambda) = \frac{N}{2}\Bigl(1-\sqrt{1-\lambda e^{2i\pi k/N}}\Bigr) \tag{1.5}$$

for the vacua $|N,0;k,0\rangle$. Equation (1.5) yields the correct classical limit $u(\lambda=0)=0$. By going through the branch cut of the square root, one joins a sheet corresponding to a different classical limit $u_{\mathrm{cl}}=N$, suitable for the vacua $|0,N;0,k\rangle$. This demonstrates the existence of the branch cuts in figure 1. Since going through a branch cut is a smooth operation, components of the parameter space connected in this way must be in the same phase. This is obviously the case for the vacua $|N,0;k,0\rangle$ and $|0,N;0,k\rangle$ that are both confining. More precisely, they are in the same oblique confining phase characterized by the integer k that represents the electric charge of the condensed dyons. At the branching points, analyticity in the vev of the chiral operator is lost. This chiral operator then overlaps with massless degrees of freedom. This is a new kind of singularity that is not associated with a singular geometry. In the example described by figure 1, both $\operatorname{tr}\phi$ and the glueball S are massless at the branching points [1]. A last subtlety is that different chiral operators vevs can have different analytic structures. For example, even if two sheets are related by a branch cut on $\mathcal{M}_{\mathrm{q}}$, the expectation value of a given chiral operator that turns out to have the same classical limit on the two sheets does not need to have any branch cut.

An interesting problem is to find general criteria for the presence of monopole singularities and/or branch cuts. A monopole that is not condensed in a vacuum $|k_1, k_2; N_1, N_2\rangle$ can become massless only if

$$k_1 - k_2 \equiv 0 \bmod N_1 \wedge N_2 \,, \tag{1.6}$$

where $N_1 \wedge N_2$ is the greatest common divisor of N_1 and N_2. This condition was derived in [1] by looking at constraints on the possible singularities of the matrix model curve. On the other hand, the presence of a branch cut relating two sheets is possible only if they are in the same phase. One can use the general analysis by 't Hooft [7], based on the Wilson and 't Hooft loop operators, to give a criterion for $|k_1, k_2; N_1, N_2\rangle$ and $|k_1', k_2'; N_1', N_2'\rangle$ to be in the same phase. As explained recently in [8], it is useful to introduce the confinement index t. It takes values in the interval $[1, N]$ and is defined to be the smallest integer such that the t^{th} tensor product of the fundamental representation of $\mathrm{SU}(N)$ does not confine. Two components can then be in the same phase only if they have the same confinement index. Within 't Hooft's classification scheme, one must have [8]

$$t = N_1 \wedge N_2 \wedge (k_1 - k_2)\,, \tag{1.7}$$

where, by adding a multiple of N if need be, $1 \leq k_1 - k_2 \leq N$. An interesting remark [8] is that components in the same phase, and in particular with the same value of t, can correspond to different classically unbroken gauge groups. This is of course possible because the notion of a broken gauge group only makes sense classically. Unfortunately, it appears that components with the same values of t are not necessarily connected, and more general criteria are needed [8].

In the present note, we derive the quantum parameter space for the gauge group U(4). Our motivation was to check on a rather complex example that the phenomena described in [1] are generic, and in particular that the results of [8] can be understood in terms of the ideas reviewed above. We will demonstrate that the eighteen components of the U(4) parameter space are all related to each other through one of the two mechanisms described above. The final result is summarized in figure 2. We also briefly discuss the effective description of the parameter space with the help of the glueball superpotentials of [4]. Finally, in the concluding section, we list a series of open problems in the field, including remarks on large N and the Dijkgraaf-Vafa matrix model description.

2 The case of $\mathrm{U}(4)$

2.1 Calculations

The calculations relevant to the ten sheets $|k,0;4,0\rangle$, $|0,k;0,4\rangle$ and $|k,k;2,2\rangle$ were already performed in [1]. Formula (1.5) shows that we have branching points at

$$\lambda = \lambda_{\mathrm{c},k} = e^{-i\pi k/2}\,, \quad 0\leq k\leq 3\,, \tag{2.1}$$

and associated branch cuts joining the sheets $|k,0;4,0\rangle$ and $|0,k;0,4\rangle$. Both $\operatorname{tr}\phi$ and the glueball superfield S are massless at the branching points and give equally valid description of the low energy physics. It turns out that there is also a massless monopole at the branching point, as is actually the case for all gauge groups $\mathrm{U}(2N)$ in the sheets with $N_1N_2=0$ [1]. Through the monopole singularity, we can go to another phase $|k,k;2,2\rangle$, reducing the confinement index from 4 to 2. More precisely, we have a massless monopole at $\lambda=\pm 1$ on the sheet $|0,0;2,2\rangle$ and a massless monopole at $\lambda=\pm i$ on the sheet $|1,1;2,2\rangle$. At this stage of the analysis, the ten disconnected sheets we started from thus appear to be grouped together in two five-sheeted connected branches, each very similar to the $\mathrm{U}(2)$ quantum parameter space depicted in the figure 3 of [1].

The remaining eight sheets correspond to the state $|0,1;2,2\rangle$, $|1,0;2,2\rangle$, $|k,0;3,1\rangle$ and $|0,k;1,3\rangle$. All those sheets have confinement index 1 and are thus in a Coulomb phase [8]. They can a priori be related to each other through branch cuts. Our general discussion suggests that this possibility is realized through the existence of branch cuts in the analytic function $u(\lambda)$ defined in (1.3). By taking into account the multiplicities due to chiral symmetry breaking in the low energy gauge groups, we expect that u will satisfy a degree eight polynomial equation $Q_8(u)=0$ with a classical polynomial

$$Q_{8,\,\mathrm{cl}}(u) = (u-1)^3(u-2)^2(u-3)^3\,. \tag{2.2}$$

The exact quantum polynomial can be straightforwardly obtained by using the results of [2, 3]. The vacua with $N_1N_2\neq 0$ are described by the following equation,

$$g^2P_+(x)P_-(x) = (x-h_1)^2(x-h_2)^2\left(x^2(m+gx)^2-4Sgx-r\right)\,, \tag{2.3}$$

where

$$P_\pm(x) = \prod_{i=1}^{4}(x - x_i) \mp 2\Lambda^4\,, \tag{2.4}$$

S is the glueball superfield and the x_i are such that

$$\operatorname{tr}\phi^q = \sum_{i=1}^{4} x_i^q\,, \quad 1 \le q \le 4\,. \tag{2.5}$$

The formula (2.3) yields eight equations for the eight unknown variables x_i, S, h_1, h_2 and r. The solutions to (2.3) corresponding to the cases where P_+, or P_-, has two double roots yield the sheets $|k,k;2,2\rangle$. We are thus interested in the other possibility, where P_+ and P_- each have one double root,

$$P_+(x) = (x-h_1)^2(x-a_1)(x-a_2)\,, \quad P_-(x) = (x-h_2)^2(x-b_1)(x-b_2)\,. \tag{2.6}$$

The matrix model curve, on which the constraint (1.6) applies, is

$$y_{\mathrm{MM}}^2 = g^2(x-a_1)(x-a_2)(x-b_1)(x-b_2) = x^2(m+gx)^2 - 4Sgx - r\,. \tag{2.7}$$

By a straightforward, but tedious, algebraic elimination of all the variables but $u = -(g/m)(x_1+x_2+x_3+x_4)$, or much more efficiently by plugging the equations in Mathematica, we get

$$Q_8(u) = (u-1)^3(u-2)^2(u-3)^3 + \frac{\lambda^4}{64} = 0\,. \tag{2.8}$$

The eight roots of (2.8), for which explicit formulas generalizing (1.5) can be given, describe the eight sheets that we consider. It is straightforward to find the small λ expansions in the various sheets. The good expansion parameter is $\lambda^{1/3}$. For example, in a given set of conventions for the integers k_i, we have

$$|0,1;2,2\rangle: \quad u = 2 + \lambda^2/8 + \cdots \tag{2.9}$$

$$|0,0;3,1\rangle: \quad u = 1 + \lambda^{4/3}/8 + 7\lambda^{8/3}/384 + 5\lambda^4/1024 + \cdots \tag{2.10}$$

$$|0,0;1,3\rangle: \quad u = 3 - \lambda^{4/3}/8 - 7\lambda^{8/3}/384 - 5\lambda^4/1024 + \cdots \tag{2.11}$$

The expansions for the other sheets are obtained by using 2π shifts of the bare θ angle, which amounts to performing the following changes,

$$|k_1,k_2;N_1,N_2\rangle \to |k_1+1,k_2+1;N_1,N_2\rangle\,, \quad \lambda^{1/3} \to e^{-i\pi/6}\lambda^{1/3}\,. \tag{2.12}$$

The fractional instanton series for u converge only for $|\lambda|^4 < 27/4$. There are branching points for the critical values

$$\lambda^{1/3} = \tilde\lambda_{\mathrm{c},k}^{1/3} = \frac{3^{1/4}}{2^{1/6}}\, e^{-i\pi k/6}, \quad 0 \le k \le 11\,, \tag{2.13}$$

at which two pairs of roots of the polynomial Q_8 in (2.8) coincide. Those pairs correspond to the pairs of sheets $(|0,0;3,1\rangle, |1,0;2,2\rangle)$ and $(|0,0;1,3\rangle, |0,1;2,2\rangle)$ for $\lambda = \tilde{\lambda}_{\mathrm{c},0}$, and the other cases are deduced from (2.12). The full analytic structure is depicted on figure 2, and shows that the eight classically disconnected vacua are fully connected in the quantum theory.

The critical points (2.13) are the exact analogues of the critical points (2.1). In particular, it is possible to show that the glueball field S is massless. This must be so because the small λ values of $\langle S\rangle$ are different on the different sheets. An explicit check can be performed by computing the algebraic equation satisfied by $\sigma = S/(m\Lambda^2)$. By eliminating all the variables but S from (2.3) we find a degree eight polynomial equation

$$R_8(\sigma) = \sigma^8 - \frac{\lambda^2}{64}\sigma^2 + \frac{\lambda^4}{1024} = 0\,. \tag{2.14}$$

The eight solutions correspond to the eight sheets, and the singularities with massless glueballs correspond to critical values of λ for which two roots of R_8 coincide. As expected, this occurs precisely for the values (2.13).

The last step in the calculation of $\mathcal{M}_{\mathrm{q}}$ consists of studying possible phase transitions with massless monopoles relating the eight Coulomb sheets discussed above to the confining sheets. By plugging $a_1 = a_2$ or $b_1 = b_2$ in equations (2.6) and (2.7), we get four values of λ with a massless monopole,

$$\lambda = \lambda_{\mathrm{monopole\,,k}} = (4/5)e^{-i\pi k/2}\,. \tag{2.15}$$

To identify precisely on which sheets the singularities appear, one can calculate u in each cases. For $\lambda = -4i/5$, we obtain $u = 2 - 2/\sqrt{5}$ or $u = 2 + 2/\sqrt{5}$. The first value is common to the sheets $|1,0;4,0\rangle$ and $|0,0;3,1\rangle$, and the second value is common to $|0,1;0,4\rangle$ and $|0,0;1,3\rangle$. All the other cases are deduced from (2.12). In particular, there is no massless monopole point on the sheets $|1,0;2,2\rangle$ and $|0,1;2,2\rangle$, a fact that also follows from (1.6).

The final outcome is a fully connnected space $\mathcal{M}_{\mathrm{q}}$ depicted on figure 2.

We could proceed to the study of higher ranks by using the same ideas. For example, the twenty Coulomb sheets of the U(5) theory are separated in two sets of ten related to each other by a 2π shift in θ. The couples of integers (N_1, N_2) in each set are $(1,4)$ and $(4,1)$ $(2+2$ states), and $(2,3)$ and $(3,2)$ $(3+3$ states). From this we deduce the classical polynomial for u, analogous to (2.2),

$$Q_{10,\,\mathrm{cl}}(u) = (u-1)^2(u-2)^3(u-3)^3(u-4)^2 = 0\,. \tag{2.16}$$

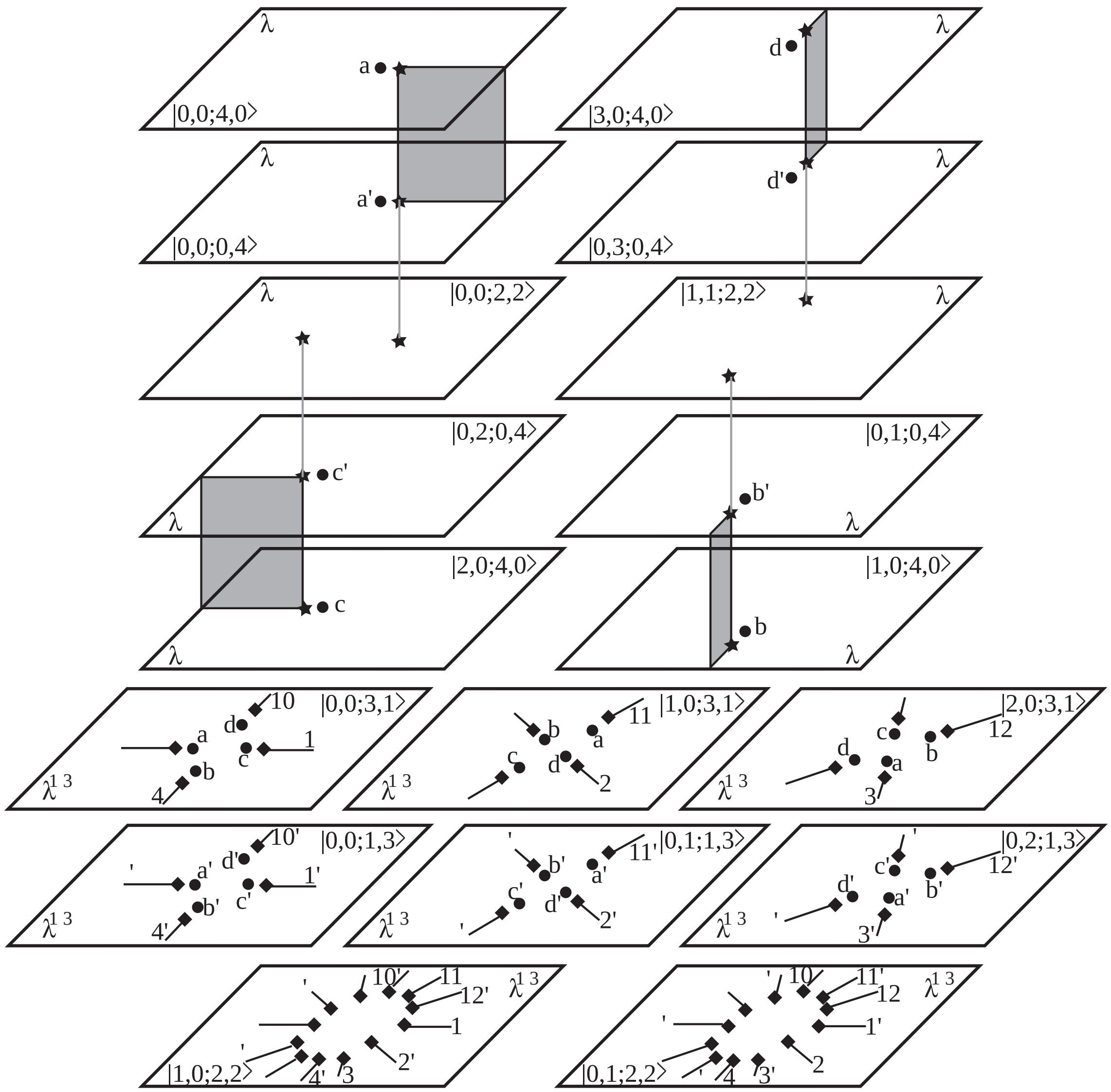

Figure 2: Sketch of the quantum parameter space $\mathcal{M}_{\mathrm{q}}$ for gauge group U(4). The sheets are parametrized by λ or $\lambda^{1/3}$. The dots, squares and stars represent singularities with a massless monopole (at $\lambda = (4/5)e^{-ik\pi/2}$), a massless glueball (at $\lambda^{1/3} = (3^{1/4}/2^{1/6})e^{-ik\pi/6}$), or both (at $\lambda = e^{-ik\pi/2}$), respectively. Due to the complexity of the diagram, we have not been able to represent explicitly all the identifications between sheets. It is understood that singularities and branch cuts with the same label are identified.

The quantum equation can be found to be

$$Q_{10}(u) = Q_{10,\,\mathrm{cl}}(u) - \frac{96g^5\Lambda^5}{m^5}(u-3)^2(u-2)^2(2u-5) - \frac{32g^{10}\Lambda^{10}}{m^{10}} = 0\,. \tag{2.17}$$

This shows that the ten sheets are related to each other through branch cuts. Phase transitions with the confining vacua can also be found straightforwardly. They occur for

$$89^5\lambda^{10} - 9281\times 19\times 2^{15}\,\lambda^5 + 2^{30} = 0 \tag{2.18}$$

and make $\mathcal{M}_{\mathrm{q}}$ fully connected.

2.2 An effective description

It is interesting to discuss the effective description in terms of the glueball superpotential $W(S_i)$. At small S_i, the calculation of W starts by choosing a classical vacuum around which one expands in terms of planar Feynman diagrams [9]. As was explained by using the matrix model in the Appendix of [1], the equations of motion $\mathrm{d}W=0$ have automatically $\prod_j N_j$ solutions. In our case, the solutions describe automatically the vacua $|k_1,k_2;N_1,N_2\rangle$ for given N_1 and N_2 and any k_1 and k_2.

A natural question is whether $W(S_i)$ is also able to describe the smooth interpolations between different sheets, that correspond to different classical limits, and thus to different integers N_j. The most basic example was studied in [1], where it was shown that the interpolation between the vacua $|k,0;N,0\rangle$ and $|0,k;0,N\rangle$ (which is a strong coupling effect in the theory with superpotential (1.1)) is correctly described by $W(S)$. We believe that this is a completely general phenomenon.[1] In the case of U(4), for example, the interpolation between the eight Coulomb components can be described by $W(S_1,S_2)$. A direct way to understand this is that the expectation values $\langle S_1+S_2\rangle = \langle S\rangle$ relevant to those components and deduced from $\mathrm{d}W=0$ must satisfy the equation (2.14) that describes the interpolation between the different sheets. Of course, this is a genuine non-perturbative effect that cannot be seen in the expansion of $W(S_i)$ in powers of S_i.

[1] This seems to contradict a claim made in [8].

3 Conclusion

For the first time it is possible to compute exactly and systematically the quantum parameter spaces for a very large class of $\mathcal{N} = 1$ supersymmetric gauge theories. This is an important step forward compared to the more conventional discussions of quantum moduli spaces, which are plagued by the presence of massless scalars. Many aspects remain to be uncovered. In the U(N) model with one adjoint Higgs on which we have focused, it would be desirable to better understand the structure and the rôle of multicritical points. There will be analogues of Argyres-Douglas points and also generalizations of the massless glueball points [1]. The study of other gauge groups and/or matter contents is likely to produce new interesting results. In particular, it is tantalizing to study chiral models.

As emphasized in [1], very interesting phenomena are also associated with the spectrum of domain walls and the large N limit. It was shown in particular that the large N expansion can break down near strong coupling singularities, providing new examples of a phenomenon first discussed in [6, 10, 11, 12, 13]. The most intriguing aspect is that it is possible to define double scaling limits, generalizing the old approach to non-critical strings [14] to the case of four dimensions [15].

At small S_i, the Dijkgraaf-Vafa matrix model is a very simple theory of D-branes [9] and open/closed string duality. However, by far the most interesting physics occurs in regimes where the perturbative approximation to the matrix model breaks down. At large S_i, a description in terms of closed strings only is no longer possible. Open strings must appear [10], and the double scaling limits considered in [11] should correspond to a continuum limit for these open strings. An explicit description of this "enhançon" mechanism [16] in terms of the matrix model would certainly provide important new insights.

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